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Translated by L. K.

DIFFRACTION OF A CYLINDRICAL HYDROACOUSTIC WAVE AT THE JOINT OF TWO SEMI-INFINITE PLATES

PMM Vol. 33, №2, 1969, pp. 240-250

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(Received July 8, 1968)

The two-dimensional stationary diffraction problem is considered. A fluid medium fills the lower half-plane in which acoustic effects are generated by a point type source located at a certain depth. The surface of the fluid is covered by two abutting semi-infinite plates. Mechanical properties of the two plates are assumed to be different. An exact mathematical solution of the problem is constructed for the case in which conditions at the plates abuttment are not fixed. This solution (which we shall call "general") contains a certain number of arbitrary constants. The method for determination of these constants for specified conditions at the joint is indicated. A characteristic of the latter problem is that formal application of the boundary contact operators to the general solution generates divergent integrals of expressions which increase algebraically at infinity.

The analysis is carried out in certain abstract terms. The expressions of boundary, and boundary contact operators are not specified, hence these results are valid for the various methods used in plate theory approximations. The derived solutions may also be used for other boundary conditions (e. g. when one part of the fluid surface is left free, or covered by a membrane).

1. Formulation of problem. A compressible fluid fills the lower half-plane $(-\infty < x < +\infty,\ 0 < y < +\infty)$. Two semi-infinite plates lie on the surface of the fluid $(y = 0)$ extending respectively in the positive and negative directions of the x-axis (Fig. 1). The field generated in the described system by a point source of harmonic oscillations (at point x_0, y_0) is to be defined. Factor $e^{-i\omega t}$ defining the dependence of processes on time will be everywhere omitted.

We shall describe the acoustic processes in the fluid in terms of pressure $P(x, y)$. The problem as stated consists of finding a solution of the inhomogeneous Helmholtz' equation

$$\Delta P + k^2 P = \delta(x - x_0,\ y - y_0) \quad (-\infty < x < +\infty,\quad 0 < y < +\infty) \qquad (1.1)$$

with boundary conditions

$$L_1 P = 0 \quad (x > 0), \qquad L_2 P = 0 \quad (x < 0) \qquad (1.2)$$

Here
$$L_\alpha P = \left[m_{\alpha 1}\left(-\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial y} + m_{\alpha_2}\left(-\frac{\partial^2}{\partial x^2}\right)\right] P(x, 0) \qquad (\alpha = 1, 2) \qquad (1.3)$$

Operators $m_{\alpha 1}$ and $m_{\alpha 2}$ are polynomials of argument $-\partial^2 \cdot /\partial x^2$.

Coefficients of these polynomials are expressed by the mechanical parameters of the problem, and generally speaking, depend mainly on the wave number k. It is assumed that the character of this dependence is subject to limitations given below [1].

Algebraic functions $l_\alpha(\lambda)$ do not have real roots on the Riemann surface basic sheet of the complex variable λ for $\mathrm{Re}k \geqslant 0$, $\mathrm{Im}k > 0$
$$l_\alpha(\lambda) = -\sqrt{\lambda^2 - k^2} m_{\alpha 1}(\lambda^2) + m_{\alpha 2}(\lambda^2) \qquad (1.4)$$

Selection of the basic sheet for $\sqrt{\lambda^2 - k^2}$ is made here and in the following in the manner that follows. From point $\lambda = k$ a branch cut is directed upwards (dotted line on Fig. 2). The contour of this cut must not pass through the roots of $l_\alpha(\lambda)$, and

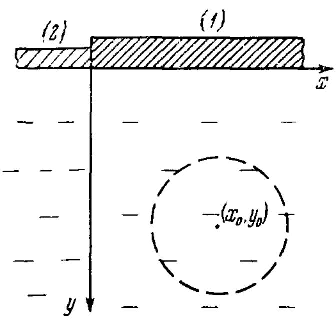

Fig. 1

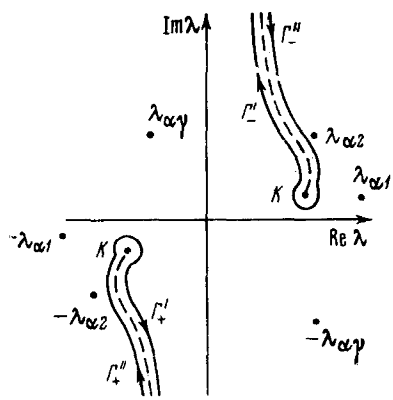

Fig. 2

is otherwise arbitrary. Branch cut through point $\lambda = -k$ is made downwards maintaining the drawing central symmetry with respect to the coordinate origin. It is assumed that on the basic sheet lim Re is
$$\sqrt{\lambda^2 - k^2} = +\infty \quad \text{for } \lambda \to \pm\infty$$
On the complimentary sheet we have correspondingly
$$\lim \mathrm{Re}\sqrt{\lambda^2 - k^2} = -\infty \quad \text{for} \quad \lambda \to \pm\infty$$

A number of examples of specific values of operators L_α is given in [1]. We shall mention only one of these pertaining to the most commonly used, viz.
$$L_\alpha = \left(\frac{\partial^4}{\partial x^4} - \frac{\mu_\alpha \omega^2}{D_\alpha}\right)\frac{\partial}{\partial y} + \frac{\rho\omega^2}{D_\alpha} \qquad (\alpha = 1, 2) \qquad (1.5)$$

Here μ_α is the plate surface density, D_α the plate cylindrical stiffness, and ρ the fluid density. Relation (1.5) corresponds to the case in which the plate lying on fluid surface is capable of flexural deformations only obeying the Kirchhoff equation. It is assumed that pressure $P(x, y) \to 0$ for $\sqrt{x^2 + y^2} \to \infty$ $(\mathrm{Im}\, k > 0,\ \mathrm{Re}\, k \geqslant 0)$ is exponential. In this solution the case of real k is considered as the transition to the limit $\mathrm{Jm}k \to +0$ (the principle of absorption limit). At the coordinate origin $P(x, y)$ is assumed to be continuous.

A solution satisfying these requirements will be in accordance with [2] called general.

It will be constructed in Sect. 2. The general solution contains n (number n is defined in Sect. 2) arbitrary constants. This violation of the requirement of uniqueness is due to the so far undefined state at the coordinate origin. We shall specify this state by n supplementary boundary contact requirements

$$R_{1\beta}P + R_{2\beta}P = 0 \qquad (\beta = 1, 2, \ldots, n) \tag{1.6}$$

$$R_{1\beta}P = \lim_{x\to+0}\left[s_{1\beta 1}\left(-i\frac{\partial}{\partial x}\right)\frac{\partial}{\partial y} + s_{1\beta 2}\left(-i\frac{\partial}{\partial x}\right)\right]P(x, 0)$$

$$R_{2\beta}P = \lim_{x\to-0}\left[s_{2\beta 1}\left(-i\frac{\partial}{\partial x}\right)\frac{\partial}{\partial y} + s_{2\beta 2}\left(-i\frac{\partial}{\partial x}\right)\right]P(x, 0) \tag{1.7}$$

Operators $s_{\alpha\beta 1}$ and $s_{\alpha\beta 2}$ are polynomials of argument $-id/dx$. In the following (Sect. 4) a certain limitation will be imposed on the algebraic properties of functions

$$r_{\alpha\beta}(\lambda) = -\sqrt{\lambda^2 - k^2}\, s_{\alpha\beta 1}(\lambda) + s_{\alpha\beta 2}(\lambda) \tag{1.8}$$

We shall adduce examples of boundary contact conditions when L_α is defined by relation (1.5). (Here $n = 4$.)

A. An infinitely narrow fissure exists between the two plates [3]

$$\lim_{x\to\pm 0}\frac{\partial^3 P(x, 0)}{\partial x^2 \partial y} = 0, \qquad \lim_{x\to\pm 0}\frac{\partial^4 P(x, 0)}{\partial x^3 \partial y} = 0 \tag{1.9}$$

B. The plates are soldered together [4]

$$\lim_{x\to+0}\frac{\partial P(x, 0)}{\partial y} = \lim_{x\to-0}\frac{\partial P(x, 0)}{\partial y}, \qquad \lim_{x\to+0}\frac{\partial^2 P(x, 0)}{\partial x \partial y} = \lim_{x\to-0}\frac{\partial^2 P(x, 0)}{\partial x \partial y} \tag{1.10}$$

$$D_1 \lim_{x\to+0}\frac{\partial^3 P(x, 0)}{\partial x^2 \partial y} = D_2 \lim_{x\to-0}\frac{\partial^3 P(x, 0)}{\partial x^2 \partial y}, \qquad D_1 \lim_{x\to+0}\frac{\partial^4 P(x, 0)}{\partial x^3 \partial y} = D_2 \lim_{x\to-0}\frac{\partial^4 P(x, 0)}{\partial x^3 \partial y} \tag{1.11}$$

C. The soldered joint of plates is reinforced by a stiffening rib [5]. In this case relations (1.10) are fulfilled, while the more complicated expressions

$$D_1 \lim_{x\to+0}\frac{\partial^3 P(x, 0)}{\partial x^2 \partial y} - D_2 \lim_{x\to-0}\frac{\partial^3 P(x, 0)}{\partial x^2 \partial y} + i\omega Z_M \lim_{x\to\pm 0}\frac{\partial^2 P(x, 0)}{\partial x \partial y} = 0$$

$$D_1 \lim_{x\to+0}\frac{\partial^4 P(x, 0)}{\partial x^3 \partial y} - D_2 \lim_{x\to-0}\frac{\partial^4 P(x, 0)}{\partial x^3 \partial y} + i\omega Z_Q \lim_{x\to\pm 0}\frac{\partial P(x, 0)}{\partial y} = 0 \tag{1.12}$$

are substituted for (1.11).

Expressions for impedence Z_M and Z_Q are given in [5].

2. Derivation of the general solution. We shall derive a solution satisfying all conditions of the problem, with the exception of the boundary contact condition.

We write the expression for field P in the form of the sum of three terms

$$P = P_0 + P_1^* + P_2^* \tag{2.1}$$

Here P_0 represents the point source field of an infinite fluid medium

$$P_0 = -\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{i\lambda(x-x_0) - \sqrt{\lambda^2-k^2}\,|y-y_0|}\frac{d\lambda}{\sqrt{\lambda^2 - k^2}} \tag{2.2}$$

Functions $P_1^*(x, y)$ and $P_2^*(x,y)$ are to be determined. We shall assume that they individually satisfy the homogeneous Helmholtz' equation, and also the conditions of continuity at the coordinate origin and of attenuation at infinity. We shall have these functions subject to boundary conditions as follows:

$$L_1(P^*{}_1 + P_0) = 0 \quad (x > 0), \qquad L_2 P_1{}^* = 0 \quad (x < 0) \tag{2.3}$$

$$L_1 P_2{}^* = 0 \quad (x > 0) \qquad L_2(P_2{}^* + P_0) = 0 \quad (x < 0) \tag{2.4}$$

It is obvious that with this all of the problem conditions will be fulfilled for the resultant field P.

We shall first determine $P_1{}^*$. We shall look for $P_1{}^*$ in the form of series expansion in the plane waves

$$P_1{}^* = \frac{1}{4\pi}\int\limits_{-\infty}^{\infty} p_1(\lambda)\, e^{i\lambda x - \sqrt{\lambda^2-k^2}y} d\lambda \tag{2.5}$$

P_1^* will then automatically satisfy the homogeneous Helmholtz' equation, and has the required behavior when $\sqrt{x^2+y^2} \to \infty$ with the appropriate $p_1(\lambda)$ and the above-mentioned (Sect. 1) selection of the branch radical $\sqrt{\lambda^2-k^2}$.

We shall require $p_1(\lambda)$ to satisfy for $\lambda \to \pm\infty$ the following estimate

$$p_1(\lambda) = o(1/\lambda^{1+\varepsilon}) \qquad (0 < \varepsilon < 1/2) \tag{2.6}$$

Condition (2.6) is sufficient for assuring the continuity of $P_1^*(x, y)$ at the coordinate origin.

Using boundary condition (2.3) and carrying formally out the differentiation under the integral sign, we derive the following integral equations:

$$\int\limits_{-\infty}^{\infty}\left[l_1(\lambda)\, p_1(\lambda) - \frac{l_1^\circ(\lambda)}{\sqrt{\lambda^2-k^2}} e^{-i\lambda x_0 - \sqrt{\lambda^2-k^2}y_0}\right] e^{i\lambda x} d\lambda = 0 \qquad (x > 0) \tag{2.7}$$

$$\int\limits_{-\infty}^{\infty} l_2(\lambda)\, p_1(\lambda)\, e^{i\lambda x} d\lambda \qquad (x < 0) \tag{2.8}$$

Here

$$l_\alpha{}^\circ(\lambda) = \sqrt{\lambda^2-k^2}\, m_{\alpha_1}(\lambda^2) + m_{\alpha_2}(\lambda^2) \tag{2.9}$$

Values of $l_\alpha(\lambda)$ of the basic sheet of the Riemann surface coincide with those of $l_\alpha(\lambda)$ on the second sheet.

Integrals in the left sides of equalities (2.7) and (2.9) will be generally divergent because the algebraic factors $l_\alpha(\lambda)$ may as the result of differentiation cause an increase of the integrand absolute value at infinity. An interpretation of similar divergent integrals is given in [2].

Equations (2.7) and (2.8) are identically satisfied when the following relationships are fulfilled

$$l_1(\lambda)\, p_1(\lambda) - \frac{l_1{}^\circ(\lambda)}{\sqrt{\lambda^2-k_2}} e^{-i\lambda x_0 - \sqrt{\lambda^2-k^2}y_0} = \Phi_1{}^+(\lambda) \tag{2.10}$$
$$l_2(\lambda)\, p_1(\lambda) = \Phi_1{}^-(\lambda)$$

in which functions Φ_1^+ (Φ_1^-) are analytical functions in the upper (lower) half-plane of the complex variable λ. Their rate of growth in the two specified planes is assumed to be not greater than exponential.

Eliminating $p_1(\lambda)$, we obtain the Riemann inhomogeneous boundary value problem [6]. The problem consists of finding two functions Φ_1^+ and Φ_1^- analytical in upper and lower half-planes respectively from the linear relationship

$$\frac{l_1(\lambda)}{l_2(\lambda)}\Phi_1{}^-(\lambda) - \Phi_1{}^+(\lambda) = \frac{l_1{}^\circ(\lambda)}{\sqrt{\lambda^2-k^2}} e^{-i\lambda x_0 - \sqrt{\lambda^2-k^2}y_0} \tag{2.11}$$

between the two fulfilled along the real axis.

We denote by n_α the highest order derivative appearing in L_α. We shall then have for $l_\alpha(\lambda)$ at infinity the following estimate:

$$l_\alpha(\lambda) = O(\lambda^{n_\alpha}) \qquad (|\lambda| \to \infty) \tag{2.12}$$

We shall represent $l_\alpha(\lambda)$ in the form of a product of two factors $l_\alpha^\pm(\lambda)$ analytical in the upper and lower half-planes, respectively,

$$l_\alpha(\lambda) = l_\alpha^+(\lambda)\, l_\alpha^-(\lambda) \tag{2.13}$$

We shall assume that the following estimate

$$l_\alpha^\pm(\lambda) = O(\lambda^{1/2 n_\alpha}) \qquad (|\lambda| \to \infty) \tag{2.14}$$

holds for $l_\alpha^\pm(\lambda)$.

The factorization method will be given in Sect. 3.

With the use of (2.13) we may rewrite (2.11) in the form

$$\frac{l_1^-(\lambda)}{l_2^-(\lambda)} \Phi_1^-(\lambda) - \frac{l_2^+(\lambda)}{l_1^+(\lambda)} \Phi_1^+(\lambda) = \frac{l_2^+(\lambda)\, l_1^\circ(\lambda)}{l_1^+(\lambda)\sqrt{\lambda^2 - k^2}} e^{-i\lambda x_0 - \sqrt{\lambda^2 - k^2}\, y_0} \tag{2.15}$$

We introduce the piecewise-analytical function $\Psi_1(\lambda)$

$$\Psi_1(\lambda) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{l_2^+(\tau)\, l_1^\circ(\tau)}{l_1^+(\tau)\sqrt{\tau^2 - k^2}} e^{-i\tau x_0 - \sqrt{\tau^2 - k^2}\, y_0} \frac{d\tau}{\tau - \lambda} \tag{2.16}$$

Its values in the upper and lower half-planes will be denoted by Ψ_1^+ and (Ψ_1^-).

According to the formulas of Sokhotski the jump

$$\Psi_1^+(\lambda) - \Psi_1^-(\lambda) = \frac{l_2^+(\lambda)\, l_1^\circ(\lambda)}{l_1^+(\lambda)\sqrt{\lambda^2 - k^2}} e^{-i\lambda x_0 - \sqrt{\lambda^2 - k^2}\, y_0} \qquad (\mathrm{Im}\lambda = 0) \tag{2.17}$$

occurs at transition through the real axis.

With (2.17) taken into account relation (2.15) may be presented in the form

$$\frac{l_1^-(\lambda)}{l_2^-(\lambda)} \Phi_1^-(\lambda) + \Psi_1^-(\lambda) = \frac{l_2^+(\lambda)}{l_1^+(\lambda)} \Phi_1^+(\lambda) + \Psi_1^+(\lambda) \qquad (\mathrm{Im}\,\lambda = 0) \tag{2.18}$$

According to the theorem of analytical continuation through the contour, the left and and right parts of equality (2.18) define a certain unique function $F_1(\lambda)$ which is analytical throughout the complex plane λ. Because of its estimated exponential rate of increase at infinity this function will be a polynomial the power of which we denote by $n - 1$

$$F_1(\lambda) = g_{n-1}^{(1)}(\lambda) \tag{2.19}$$

The following expression of

$$n = n_1 + n_2 - E\left(\tfrac{1}{2}(n_1 + n_2 - 1)\right) \tag{2.20}$$

is readily obtained from (2.6), (2.12) and (2.13).

Symbol $E(\xi)$ denotes the whole part of number ξ. The coefficients of polynolial $q_{n-1}^{(1)}(\lambda)$ are arbitrary. Their number must coincide with the number of linearly independent boundary contact conditions.

As an illustration we adduce examples of determination of the number of boundary contact conditions from the order of boundary operators differentials.

1) Two plates lie on the surface of the fluid (flexural oscillations only are taken into consideration; L_1 and L_2 are defined by Formula (1.5))

$$n_1 = 5, \quad n_2 = 5, \quad n = 4$$

2) The fluid surface is free ($L_2 = 1$) on one side of the coordinate origin, and covered

by a plate on the other

$$n_1 = 5, \quad n_2 = 0, \quad n = 2$$

3) A rigid boundary abuts on the plate ($L_2 = \partial / \partial y$)

$$n_1 = 5, \quad n_2 = 1, \quad n = 2$$

We revert to the solution of the problem. Taking into consideration (2.5), (2.10) and (2.19) we derive the following expression for $P_1^*(x, y)$:

$$P_1^* = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{q_{n-1}^{(1)}(\lambda) - \Psi_1^-(\lambda)}{l_2^+(\lambda)\, l_1^-(\lambda)} e^{i\lambda x - \sqrt{\lambda^2 - k^2}\, y} d\lambda \tag{2.21}$$

Function $\Psi_1^-(\lambda)$ is defined by Formula (2.16), and it is assumed that the integration contour passes above pole $\tau = \lambda$.

The problem is solved in a similar manner for $P_2^*(x,y)$

$$P_2^* = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{q_{n-1}^{(2)}(\lambda) + \Psi_2^+(\lambda)}{l_2^+(\lambda)\, l_1^-(\lambda)} e^{i\lambda x - \sqrt{\lambda^2 - k^2}\, y} d\lambda \tag{2.22}$$

$$\Psi_2(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{l_1^-(\tau)\, l_2^\circ(\tau)}{l_2^-(\tau) \sqrt{\tau^2 - k^2}} e^{i\tau x_0 - \sqrt{\tau^2 - k^2}\, y_0} \frac{d\tau}{\tau - \lambda} \tag{2.23}$$

When deriving $\Psi_2^+(\lambda)$ in (2.23) the pole $\tau = \lambda$ is bypassed from below. Combining the solution of these two problems we obtain the following final expression of P:

$$P = P_0 + P_1 + P_2 + Q \tag{2.24}$$

Here

$$P_1 = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\Psi_1^-(\lambda)}{l_2^+(\lambda)\, l_1^-(\lambda)} e^{i\lambda x - \sqrt{\lambda^2 - k^2}\, y} d\lambda \tag{2.25}$$

$$P_2 = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\Psi_2^+(\lambda)}{l_2^+(\lambda)\, l_1^-(\lambda)} e^{i\lambda x - \sqrt{\lambda^2 - k^2}\, y} d\lambda \tag{2.26}$$

$$Q = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{q_{n-1}(\lambda)}{l_2^+(\lambda)\, l_1^-(\lambda)} e^{i\lambda x - \sqrt{\lambda^2 - k^2}\, y} d\lambda \tag{2.27}$$

The reflection diffracted fields P_1 and P_2 satisfy the boundary conditions (2.3) and (2.4) and have at the coordinate origin continuous derivatives up to the $(n-1)$th order.

The diffracted field Q satisfies the homogeneous boundary conditions

$$L_1 Q = 0 \quad (x > 0), \qquad L_2 Q = 0 \quad (x < 0) \tag{2.28}$$

and carries in it discontinuities of derivatives of the total field P at the coordinate origin. We note that the form of expression of Q (2.27) is independent of the character of the incident field P_0. The same expression of Q may, incidentally, be derived e.g. with the construction of the plane wave diffraction problem [2].

3. Factorization of $l_\alpha(\lambda)$. We shall assume that the power of polynomial $m_{\alpha 1}(\lambda^2)$ which shall be denoted by $2\nu_\alpha$ is not smaller than that of polynomial $m_{\alpha 2}(\lambda^2)$. Thus term $\sqrt{\lambda^2 - k^2}\, m_{\alpha 1}(\lambda^2)$ defines the behavior of $l_\alpha(\lambda)$ at infinity, and we have the following equality:

$$n_\alpha = 2\nu_\alpha + 1 \tag{3.1}$$

The case of the even number n_α could be considered in a manner similar to that given in the following. Our selection is related to that the situation defined by (3.1) is more realistic. If one disregards the case of the free surface ($L_2 = 1$, $n_2 = 0$) in which factorization is trivial ($l_2^{\pm}(\lambda) = 1$), then in the usually applicable boundary conditions the order of the differential operators L_α proves to be an odd number.

Function $l_\alpha(\lambda)$ has on the two-sheet Riemann surface $4\nu_\alpha + 2$ roots differing pairwise as to their signs. We denote these roots by $\pm\lambda_{\alpha\gamma}$ ($\gamma = 1, 2, \dots, 2\nu_\alpha + 1$) with the plus sign assigned to roots in the upper half-plane. Let us assume that there are N_α pairs of roots $l_\alpha(\lambda)$ on the basic sheet which we shall number from 1 to N_α.

Without loss of generality it may be assumed that the coefficient at the highest power of λ in $m_{\alpha 1}(\lambda^2)$ is equal to unity. We introduce functions $\varphi_\alpha(\lambda)$ related to $l_\alpha(\lambda)$ by the expression

$$l_\alpha(\lambda) = -\varphi_\alpha(\lambda)(\lambda^2 - k^2)^{\nu_\alpha + \frac{1}{2} - N_\alpha} \prod_{\gamma=1}^{N_\alpha} (\lambda^2 - \lambda_{\alpha\gamma}^2) \tag{3.2}$$

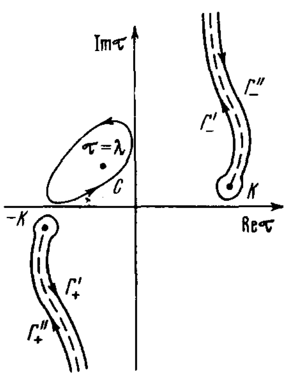

Fig. 3

Functions $\varphi_\alpha(\lambda)$ have no roots on the Riemann surface basic sheet, and tend to unity when $|\lambda| \to \infty$. Hence on this sheet $\ln\varphi_\alpha(\lambda)$ can be uniquely defined on the basis of the requirement

$$\lim_{|\lambda|\to\infty} \ln \varphi_\alpha(\lambda) = 0 \tag{3.3}$$

We represent $\ln\varphi_\alpha(\lambda)$ by the Cauchy integral

$$\ln \varphi_\alpha(\lambda) = \frac{1}{2\pi i} \int_C \frac{\ln \varphi_\alpha(\tau)}{\tau - \lambda} d\tau \tag{3.4}$$

Here C is an arbitrary contour on the day sheet of plane τ circumscribing point $\tau = \lambda$ in the positive direction (Fig. 3). By stretching this contour we can obtain the following expression:

$$\varphi_\alpha(\tau) = \varphi_\alpha^+(\tau)\varphi_\alpha^-(\tau), \qquad \ln \varphi_\alpha^{\pm}(\lambda) = \frac{1}{2\pi i} \int_{\Gamma^{\pm}} \frac{\ln \varphi_\alpha(\tau)}{\tau - \lambda} d\tau \tag{3.5}$$

Here $\Gamma_{\pm} = \Gamma_{\pm}' + \Gamma_{\pm}''$ are the contours enveloping the upper and lower branch cuts respectively in the τ-plane. By virtue of (3.3) the integrals (3.5) are convergent. For the same reason the integrals taken along arcs connecting contours Γ_+ and Γ_- vanish at the limit.

Functions φ_α^+ and φ_α^- are analytical outside the contours along which they are defined by these integrals. Function $\varphi_\alpha^+(\lambda)$ is analytical outside the branch cut extending downwards from point $\lambda = -k$, and in particular in the upper half-plane. Similarly $\varphi_\alpha^-(\lambda)$ is analytical outside the contour Γ_-. At infinity all these tend to unity.

Integrals along contours $\Gamma_{\pm}$ may be reduced to integrals along one of the edges of the branch cut

$$\ln \varphi_\alpha^{\pm}(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_{\pm}'} \ln \left[-\frac{l_\alpha(\tau)}{l_\alpha^\circ(\tau)} \right] \frac{d\tau}{\tau - \lambda} \tag{3.6}$$

Here Γ_+' denotes the left-hand edge of the branch cut drawn from point $\tau = -k$, and Γ_-' the right-hand edge of the branch cut at $\tau = k$. Both contours are traversed from points $\tau = \pm k$ towards infinity. Opposite edges of branch cuts are denoted by symbol $\Gamma_{\pm}''$. The branch of the integrand logarithm in (3.6) is selected so as to have

$$\lim_{|\tau|\to\infty} \ln \left[-\frac{l_\alpha(\tau)}{l_\alpha^\circ(\tau)} \right] = 0 \qquad (\tau \in \Gamma_{\pm}') \tag{3.7}$$

Factorization of coefficients of $\varphi_\alpha(\lambda)$ in (3.2) is achieved in an elementary way. As the result we have

$$l_\alpha^\pm(\lambda) = \pm(\lambda \pm k)^{\nu_\alpha + 1/2 - N_\alpha} \prod_{\gamma=1}^{N_\alpha} (\lambda \pm \lambda_{\alpha\gamma}) \exp\left\{\frac{1}{2\pi i} \int_{\Gamma_\pm'} \ln\left[-\frac{l_\alpha(\tau)}{l_\alpha^\circ(\tau)}\right] \frac{d\tau}{\tau - \lambda}\right\} \tag{3.8}$$

Functions $l_\alpha^\circ(\lambda)$ can be factorized in the same manner

$$l_\alpha^\circ(\lambda) = l_\alpha^{\circ+}(\lambda)\, l_\alpha^{\circ-}(\lambda) \tag{3.9}$$

$$l_\alpha^{\circ\pm}(\lambda) = (\lambda \pm k)^{N_\alpha - \nu_\alpha - 1/2} \prod_{\gamma=N_\alpha+1}^{2\nu_\alpha+1} (\lambda \pm \lambda_{\alpha\gamma}) \exp\left\{-\frac{1}{2\pi i} \int_{\Gamma_\pm''} \ln\left[-\frac{l_\alpha(\tau)}{l_\alpha^\circ(\tau)}\right] \frac{d\tau}{\tau - \lambda}\right\} \tag{3.10}$$

We write down some of the relationships useful in the various transformations of the solution obtained in Sect. 2

$$\begin{aligned} l_\alpha(\lambda)\, l_\alpha^\circ(\lambda) &= -\prod_{\gamma=1}^{2\nu_\alpha+1} (\lambda^2 - \lambda_{\alpha\gamma}^2), \qquad & l_\alpha^+(-\lambda) &= e^{\pi i(\nu_\alpha - 1/2)}\, l_\alpha^-(\lambda) \\ l_\alpha^\pm(\lambda)\, l_\alpha^{\circ\pm}(\lambda) &= \pm \prod_{\gamma=1}^{2\nu_\alpha+1} (\lambda \pm \lambda_{\alpha\gamma}), \qquad & l_\alpha^{\circ+}(-\lambda) &= e^{\pi i(\nu_\alpha + 1/2)}\, l_\alpha^{\circ-}(\lambda) \end{aligned} \tag{3.11}$$

The first of these is obtained in an elementary way, while the remaining three follow from Formulas (3.8) and (3.10).

By virtue of the Sokhotski's formulas it is also possible to obtain for the Cauchy type integrals in (3.8) and (3.10) the bypass relationsips for functions $l_\alpha^\pm(\lambda)$ and $l_\alpha^{\circ\pm}(\lambda)$

$$l_\alpha^\pm(\lambda)\big|_{\lambda\in\Gamma_\pm''} = \frac{l_\alpha^\circ(\lambda)}{l_\alpha(\lambda)}\, l_\alpha^\pm(\lambda)\Big|_{\lambda\in\Gamma_\pm'} \qquad l_\alpha^{\circ\pm}(\lambda)\big|_{\lambda\in\Gamma_\pm''} = \frac{l_\alpha(\lambda)}{l_\alpha^\circ(\lambda)}\, l_\alpha^{\circ\pm}(\lambda)\Big|_{\lambda\in\Gamma_\pm'} \tag{3.12}$$

4. Boundary contact conditions. The direct application to field P of the boundary contact operators $R_{\alpha\beta}$ results in the known difficulty in that it generally produces divergent integrals of expressions which increase algebraically at infinity. It will be shown in the following how to express $R_{\alpha\beta}P$ in terms of integrals convergent in the conventional sense. This will necessitate the introduction of certain limiting assumption as regards operators $R_{\alpha\beta}$

We separate from P the incident wave P_0 and wave P_{01} reflected from the right-hand plate, and readily obtain the expressions

$$P = P_0 + P_{01} + Q_1 \tag{4.1}$$

$$P_{01} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{l_1^\circ(\lambda)}{\sqrt{\lambda^2 - k^2}\, l_1(\lambda)} e^{i\lambda(x - x_0) - \sqrt{\lambda^2 - k^2}(y + y_0)}\, d\lambda \tag{4.2}$$

$$Q_1 = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{q_+(\lambda)}{l_2^+(\lambda)\, l_1^-(\lambda)} e^{i\lambda x - \sqrt{\lambda^2 - k^2}\, y}\, d\lambda \tag{4.3}$$

Here

$$q_+(\lambda) = q_{n-1}(\lambda) - \Psi_1^+(\lambda) + \Psi_2^+(\lambda) \tag{4.4}$$

represents a function which is analytical in the upper half-plane.

The direct application of $R_{1\beta}$ to P_0 and P_{01} does not represent any difficulty, as the transition to limit $y = 0$, $x \to +0$ there remains in the integrand the exponential factor $\exp(-\sqrt{\lambda^2 - k^2}\, y_0)$ which ensures convergence of the integral.

We now turn to $R_{1\beta}Q_1$. We have

$$R_{1\beta}Q_1 = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{r_{1\beta}(\lambda)\, q_+(\lambda)}{l_2^+(\lambda)\, l_1^-(\lambda)}\, e^{+i0\lambda}\, d\lambda \tag{4.5}$$

The following notation is used here

$$\lim_{x\to\pm 0}\int_{-\infty}^{\infty} f(\lambda)\, e^{i\lambda x}\, d\lambda = \int_{-\infty}^{\infty} f(\lambda)\, e^{\pm i0\lambda}\, d\lambda \tag{4.6}$$

We shall distort (stretch) the integration contour into the upper half-plane until it reaches position Γ. With this the integrand poles lying in the roots of $l_1^-(\lambda)$ will be crossed

$$R_{1\beta}Q_1 = \frac{i}{2}\sum_{\operatorname{Im}\lambda>0} \operatorname{Res}\frac{r_{1\beta}(\lambda)\, q_+(\lambda)}{l_2^+(\lambda) l_1^-(\lambda)} + \frac{1}{4\pi}\int_{\Gamma_-} \frac{r_{1\beta}(\lambda)\, q_+(\lambda)}{l_2^+(\lambda)\, l_1^-(\lambda)}\, e^{+i0\lambda}\, d\lambda \tag{4.7}$$

Using the bypass relationships (3.12) we can substitute the integral along one of the edges of the branch cut

$$R_{1\beta}Q_1 = \frac{i}{2}\sum_{\operatorname{Im}\lambda>0} \operatorname{Res}\frac{r_{1\beta}(\lambda)\, q_+(\lambda)}{l_2^+(\lambda)\, l_1^-(\lambda)} +$$

$$+\frac{1}{4\pi}\int_{\Gamma_-} q_+(\lambda)\frac{l_1^+(\lambda)}{l_2^+(\lambda)}\, \frac{r_{1\beta}(\lambda)\, l_1^\circ(\lambda) - r_{1\beta}^\circ(\lambda)\, l_1(\lambda)}{l_1^\circ(\lambda)\, l_1(\lambda)}\, e^{+i0\lambda}\, d\lambda \tag{4.8}$$

Here

$$r_{\alpha\beta}^0(\lambda) = \sqrt{\lambda^2 - k^2}\, s_{\alpha\beta 1}(\lambda) + s_{\alpha\beta 2}(\lambda) \tag{4.9}$$

for the integral along Γ_-.

We impose on $R_{\alpha\beta}$ the following limitation:

$$r_{\alpha\beta}(\lambda)\, l_\alpha^\bullet(\lambda) - r_{\alpha\beta}^\bullet(\lambda)\, l_\alpha(\lambda) = O(\lambda^{n_\alpha}) \qquad (|\lambda|\to\infty) \tag{4.10}$$

In other words the rate of growth of function $|r_{\alpha\beta}(\lambda)\, \dot{l}_\alpha(\lambda) - r_{\alpha\beta}^\circ(\lambda)\, l_\alpha(\lambda)|$ at infinity is not greater than $|l_\alpha(\lambda)|$.

This limitation establishes a certain necessary link between the boundary contact operators $R_{\alpha\beta}$ and the corresponding (i.e. pertaining to the same value of α) boundary operators L_α. The need for a relationship linking these operators is from the physical point of view quite natural. In any specific problem with a given L_α there is only a very limited choice of acceptable values of $R_{\alpha\beta}$. Relation (4.10) establishes only the simplest necessary (but not sufficient) limitation on the range of admissible values of $R_{\alpha\beta}$. A direct check will readily show that (4.10) hold for all examples adduced in Sect. 1.

Expression $r_{1\beta}(\lambda)\, l_1^\circ(\lambda) - r_{1\beta}^\circ(\lambda)\, l_1(\lambda)$ changes its sign when passing from the left-hand side edge of branch cut (Γ_-') to the right-hand one (Γ_-''). The remaining factors of the integrand (with (3.10) taken into consideration) remain unchanged when passing around point $\lambda = k$. Hence the integral along Γ_-' in (4.8) is equal to half the integral along the complete loop Γ_- of the same expression. We revert to the integration along the real axis, and finally obtain the expression

$$R_{\alpha 1}Q_1 = \frac{i}{4}\sum_{\operatorname{Im}\lambda>0} \operatorname{Res}\frac{q_+(\lambda)\, l_1^+(\lambda)}{l_2^+(\lambda)}\, \frac{r_{1\beta}(\lambda)\, l_1^\circ(\lambda) - r_{1\beta}^\circ(\lambda)\, l_1(\lambda)}{l_1(\lambda)\, l_1^\circ(\lambda)} +$$

$$+\frac{1}{8\pi}\int_{-\infty}^{\infty} \frac{q_+(\lambda)\, l_1^+(\lambda)}{l_2^+(\lambda)}\, \frac{r_{1\beta}(\lambda)\, l_1^\circ(\lambda) - r_{1\beta}^\circ(\lambda)\, l_1(\lambda)}{l_1(\lambda)\, l_1^\circ(\lambda)}\, d\lambda \tag{4.11}$$

In the first term summation extends along roots $l_1(\lambda)$ and $l_1^\circ(\lambda)$ (see (3.11)) lying in the upper half-plane.

By virtue of (4.10) the integral in (4.11) is absolutely convergent, hence the transition to limit in (4.11) has been carried out and factor $e^{+i0\lambda}$ omitted.

The transition to limit $x \to -0$ in $R_{\alpha\beta} P$ is analyzed in a similar manner. In this case field P is first converted to the form

$$P = P_0 + P_{02} + Q_2 \tag{4.12}$$

$$P_{02} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{l_2^\circ(\lambda)}{\sqrt{\lambda^2 - k^2}\, l_2(\lambda)} e^{i\lambda(x - x_0) - \sqrt{\lambda^2 - k^2}(y + y_0)}\, d\lambda \tag{4.13}$$

$$Q_2 = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{q_-(\lambda)}{l_2^+(\lambda)\, l_1^-(\lambda)} e^{i\lambda x - \sqrt{\lambda^2 - k^2} y}\, d\lambda \tag{4.14}$$

$$q_-(\lambda) = q_{n-1}(\lambda) - \Psi_1^-(\lambda) + \Psi_2^-(\lambda) \tag{4.15}$$

It then appears that

$$R_{2\beta} Q_2 = -\frac{i}{4} \sum_{\operatorname{Im}\lambda < 0} \operatorname{Res} \frac{q_-(\lambda)\, l_2^-(\lambda)}{l_1^-(\lambda)} \frac{r_{2\beta}(\lambda)\, l_2^\circ(\lambda) + r_{2\beta}^\circ(\lambda)\, l_2(\lambda)}{l_2(\lambda)\, l_2^\circ(\lambda)} +$$
$$+ \frac{1}{8\pi} \int_{-\infty}^{\infty} \frac{q_-(\lambda)\, l_2^-(\lambda)}{l_1^-(\lambda)} \frac{r_{2\beta}(\lambda)\, l_2^\circ(\lambda) - r_{2\beta}^\circ(\lambda)\, l_2(\lambda)}{l_2(\lambda)\, l_2^\circ(\lambda)}\, d\lambda \tag{4.16}$$

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Translated by J. J. D.